Delay-induced chaos in a traffic flow model

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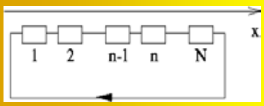
We study the presence of chaos in a car-following model based on a system of delay-differential equations. We find that for low and high values of cars density the system has a stable steady state solution. For sufficiently large delay the system may pass to chaos according to Ruelle-Takens-Newhouse scenario (fixed point – limit cycles – two-tori – three-tori – chaos) with changing the density towards intermediate values. Exponential decay of the power spectrum and non-integer correlation dimension suggest the existence of chaos. The attractor has multi-fractal properties.

1. Model introduction



N cars move in a circuit of length *L* and each car's acceleration is defined by the delay-differential equation:

$$\frac{d^2x(t)}{dt^2} = A\left(1-\frac{\Delta x_n^0(t-\tau)}{\Delta x_n(t-\tau)}\right) - \frac{Z^2(-\Delta v_n(t-\tau))}{2(\Delta x_n(t-\tau)-D)} - kZ(v_n(t-\tau)-v_{per}) \qquad (1)$$

x_n - car's coordinate, v_n - velocity, A, k – sensitivity parameters, D – minimal distance, v_{per} - permitted velocity, T - safety time gap, $\Delta x_n^0 = v_n T + D$ - safety distance, τ - time delay, $\Delta x_n = x_{n+1} - x_n$, $n=1, \dots, N$.

First term – to keep a time gap from the car ahead, **second** – to brake if the car head is much slower, **third** – not to exceed permitted velocity.

2. Linear stability analysis

Eqs. (1) have the **homogeneous flow solution** $v_1 = \dots = v_N = v^0$, $x_n = v^0 + (n-1)/\rho$, where $\rho = N/L$ is the flow density.

Linearizing Eqs. 1 near this solution, we have the equation

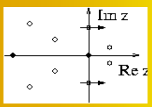
$$\ddot{\xi}_n(t) = -p(\rho)\dot{\xi}_n(t-\tau) + q(\rho)\big(\xi_{n+1}(t-\tau) - \xi_n(t-\tau)\big),$$

where $\xi_n = x_{n+1} - x_n - 1/\rho$. Looking for its solution in the form $\xi_n = \exp(i\alpha_\kappa n + zt)$, where $\alpha_\kappa = 2\pi\kappa/N$ and $\kappa = 0, \dots, N-1$, we have a set of equations for eigenvalues of system (1):

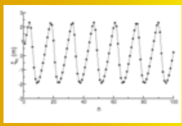
$$z^2 + [p(\rho)z - q(\rho)(e^{i\alpha_\kappa} - 1)]e^{-z\tau} = 0.$$

We find that **for low and high values of ρ** all solutions of these equations (except for one zero solution) have negative real part, which means that the **homogeneous flow solution is stable**.

3. Hopf bifurcations and limit cycles



As the density changes towards intermediate values, pairs of solutions of Eqs. 2 (for different κ) cross the imaginary axis and the homogeneous flow solution becomes unstable. The system undergoes Hopf bifurcations and new flow regimes, which are associated with limit cycles born at this bifurcations, emerge.



Any of these regimes is a wave-like flow with the wavelength equal to N/κ cars. The picture on the left corresponds to the cycle with $\kappa=7$.

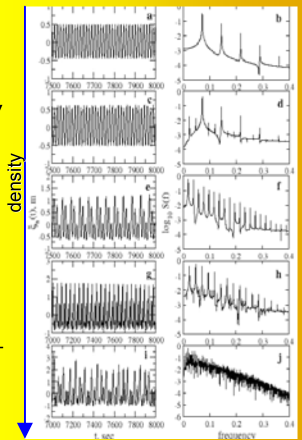
We find the cycles analytically for densities close to bifurcational values and continue them numerically changing the density.

4. Transition to chaos

For sufficiently large delay the system may pass to **chaos** with decrease of density. The cycles bifurcate into two-dimensional tori, which later bifurcate into three-dimensional tori. The motion on three-dimensional tori becomes chaotic after a finite interval of time.

The figure shows the transition to chaos from the limit cycle with $\kappa=15$.

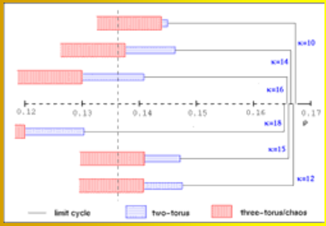
Left column – time series from a single car ($\xi_n(t)$) for decreasing density values, right column – corresponding power spectra. **a**, **b** – limit cycle; **c**, **d** – bifurcation into two-torus; **e**, **f** – developed two-torus; **g**, **h** – bifurcation into three-torus; **i**, **j** – chaos.



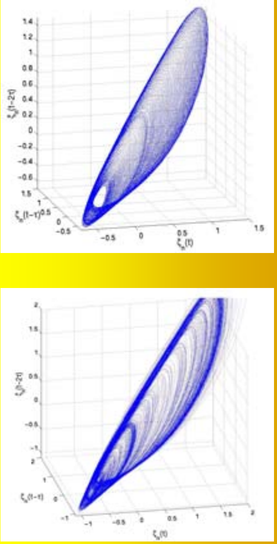
The motion on a three-torus is unstable and is driven to chaos by a small perturbation.

5. Bifurcational diagram

The biufrcational diagram shows the transition to chaos from limit cycles with κ=10, 12, 14, 15, 16 and 18. It shows that chaotic and non-chaotic flow regimes coexist for the same parameter values.



6. Time-delay reconstructions

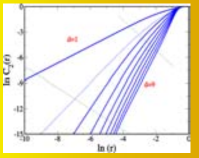


Three-dimensional time-delay reconstructions of a two-torus and a chaotic attractor.

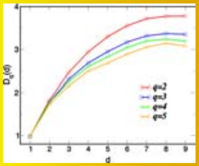
7. Correlation dimension measurements

We performed measurements of the **correlation dimension** of the attractor (for some fixed parameter values).

The first figure presents plots of the correlation function for different embedding dimensions *d*. With growing *d* the plots converge to a straight line with the slope close to **3.75**, which is the correlation dimension D_2.



The second figure presents the results of measurement of the generalized correlation dimensions $D_2, \dots, D_5$. These dimensions are different, which indicates **multi-fractality** of the attractor.



8. Publications

1. E. Tomer, L. Safonov, and S. Havlin, Phys. Rev. Lett. **84**, 382 (2000).
2. L. Safonov, E. Tomer, V. Strygin and S. Havlin, Physica A **285**, 147 (2000).
3. L. Safonov, E. Tomer, V. Strygin, Y. Ashkenazy, and S. Havlin, Europhys. Lett. (submitted)