Scale-free Networks on Lattices

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THE MODEL

Consider a d-dimensional lattice, of side R, with periodic boundary conditions.

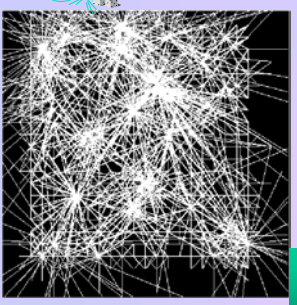
Assign random connectivity k to each site, taken from the *scale-free* distribution $P(k)=Ck^{-\lambda}$ where $m<k<K$ and $C\approx(\lambda-1)\,m^{\lambda-1}$

Select a site at random and connect it to its closest neighbors until its (previously assigned) connectivity k is realized, or until all sites up to a distance $r_{max}(k)=Ak^{1/d}$ have been explored (Links to some of the neighboring sites might prove impossible, in case that the connectivity quota of the target site is already filled.)

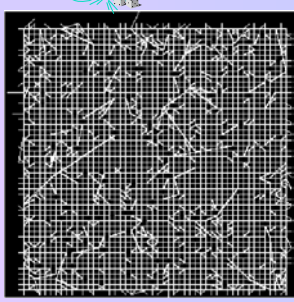
Repeat until all sites in the lattice have been selected

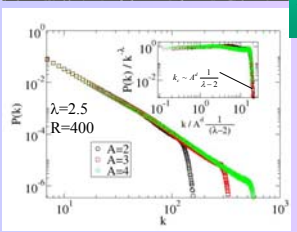
Did you ever tried to embed a *scale-free* network in a d-dimensional lattice????

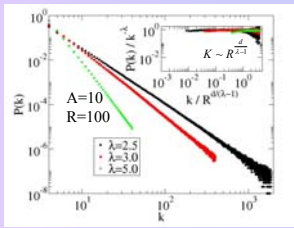
$\lambda = 2.5$



$\lambda = 5$



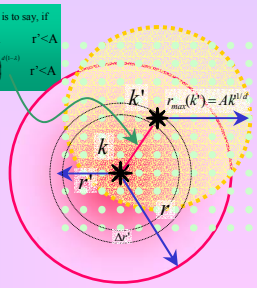




Consider the number of links $n(r)$ entering a generic site from a surrounding neighborhood of radius r.

Will be linked if $r'<r_{max}(k')$, that is to say, if

$$P\left(k'>\left(\frac{r'}{A}\right)^d\right)=C\int_{\left(\frac{r'}{A}\right)^d}^{\infty}k'^{-\lambda}dk'\sim\begin{cases}1 & r'<A\\ \left(\frac{r'}{A}\right)^{d(1-\lambda)} & r'<A\end{cases}$$



$$n(r)\sim\int_0^r dr'\,r'^{d-1}\,P\left(k'>(r'/A)^d\right)$$

$$\sim\frac{\lambda-1}{d(\lambda-2)}A^d-\frac{A^{d(\lambda-1)}}{d(\lambda-2)}r^{d(\lambda-2)}$$

The cutoff connectivity (for $\lambda>2$) is then

$$k_c=\lim_{r\to\infty}n(r)\sim\frac{1}{\lambda-2}A^d$$

Which implies a cutoff length

$$\xi=r(k_c)\sim\frac{1}{(\lambda-2)^{1/d}}A^2$$

The embedded network is *scale-free* up to distances $r<\xi$, and repeats itself (statistically) for $r>\xi$, similar to the infinite percolation cluster above criticality: The infinite cluster in percolation is *fractal* up to the coherence length ξ and repeats thereafter.

When the lattice is finite, $R<\infty$, the number of sites is finite, $N\sim R^d$. This imposes a maximum connectivity K which can be estimated from

$$\int_K^{\infty}P(k)dk=\frac{1}{N}\quad\Rightarrow\quad K\sim mN^{1/(\lambda-1)}\sim R^{d/(\lambda-1)}$$

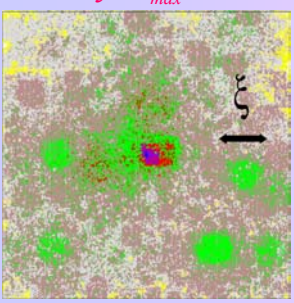
This implies a finite-size cutoff length

$$r_{max}=r(K)\sim AR^{1/(\lambda-1)}$$

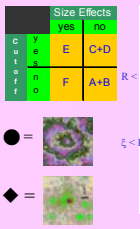
The interplay between the three length scales, R, ξ and r_{max}, determines the nature of the network. If the lattice is finite, then the maximal connectivity is $k_{max}=K$ only if $r_{max}<\xi$. Otherwise ($r_{max}>\xi$) the lattice repeats itself at length scales larger than ξ. As long as $\min(r_{max},\xi)<<R$, the finite size of the lattice imposes no serious restrictions. Otherwise ($\min(r_{max},\xi)>R$) finite-size effects become important. We emphasize that in all cases the degree distribution (up to the cutoff) is scale-free.

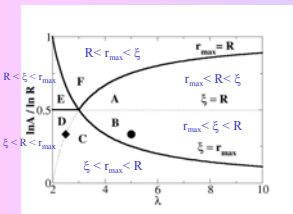
Chemical shell l consists of all sites at minimal distance (minimal number of connecting links) l from a given site.

$\xi < r_{max}$

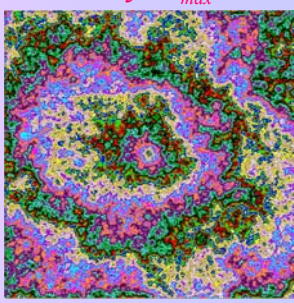


Successive chemical shells for λ=2.5





$\xi > r_{max}$



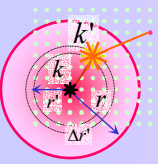
Successive chemical shells for λ=5

To study d_f, we compute the perimeter S(r), the number of sites that connect the interior cluster of a region of radius r to sites outside.

The fractal dimension d_f then follows from the scaling relation

$$S(r)\sim r^{d_f-1}$$

Consider a shell dr', of radius r'.



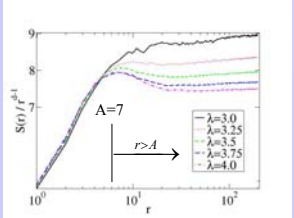
A site of connectivity k' within the shell is connected to the ouside (to a distance larger than r-r') with probability

$$P\left(k'>\left(\frac{r-r'}{A}\right)^d\right)$$

$$S(r)=\int_0^r dr'\,r'^{d-1}\,P\left(k'>\left(\frac{r-r'}{A}\right)^d\right)$$

$$\sim\begin{cases}r^d & r<A\\ c(\lambda)Ar^{d-1} & r>A\end{cases}$$

where $c(\lambda)\sim 1+1/[d(\lambda-1)+1]$. In other words, the network is compact, $d_f=d$ at large distances $r>A$, and super-compact, $d_f=d+1$, at $r<A$.



The slight slope observed for $r>A$ is due to analytical corrections, of order r^{-1}.

We now address the geometrical properties of the networks, arising from their embedding in Euclidean space. To this aim, it is useful to consider the spatial arrangement of the networks as measured both in an Euclidean metric and in *chemical space*. The chemical distance l between any two sites is the length of the minimal path between them (*minimal* number of links). Thus if the distance between the two sites is r, then $l\sim r^{d_{min}}$ defines the minimal length exponent d_{min}. We will see that $d_{min}<1$ (for $d>1$), contrary to all naturally occurring fractals and disordered media. Sites at chemical distance l from a given site constitute its l-th chemical shell. The number of (connected) sites within radius r scales as $m(r)\sim r^{d_f}$, defining the fractal dimension d_f. Likewise, the number of (connected) sites within chemical radius l scales as $m(l)\sim l^{d_l}$, which defines the fractal dimension d_l in chemical space. The two fractal dimension are related: $d_{min}=d_f/d_l$.

d_f — We focus on the regime $\xi > r_{max}$ — **d_{min}**

In order to compute d_{min} (or d_l), we regard the chemical shells as being roughly smooth, at least in the regime $\xi > r_{max}$. Let the width of shell l be $\Delta r(l)$, then, since $\Delta l=1$

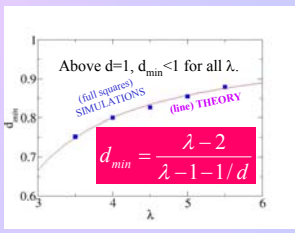
$$l=\int dl=\int\frac{dr}{\Delta r(l)}\sim r^{d_{min}}$$

The number of sites in shell l, $N(l)$, is, on the one hand, $N(l)\sim r(l)^{d-1}\Delta r(l)$. On the other hand, since the maximal connectivity in shell l is $K(l)\sim N(l)^{1/(\lambda-1)}$, the thickness of shell $(l+1)$ is $\Delta r(l+1)$ which is determined by the length of the largest link to the next shell i.e., $r(K(l))$, and thus, $\Delta r(l+1)\sim r(K(l))\sim AK(l)^{1/d}$. Assuming (for large l) that $\Delta r(l+1)\sim\Delta r(l)$, we obtain

$$\Delta r(l)\sim r^{\frac{d-1}{d(\lambda-1)-1}}$$

SUMMARY

We suggest a method for embedding scale-free networks, with degree distribution $P(k)\sim k^{-\lambda}$, in regular Euclidean lattices. The embedding is driven by a natural constraint of minimization of the total length of the links in the system. We find that all networks with $\lambda>2$ can be successfully embedded up to an (Euclidean) distance ξ which can be made as large as desired upon the changing of an external parameter. Clusters of successive chemical shells are found to be compact (the fractal dimension is $d_f=d$), while the dimension of the shortest path between any two sites is smaller than one: $d_{min}=(\lambda-2)/(\lambda-1-1/d)$, contrary to all other known examples of fractals and disordered lattices.



$$N(l)\sim l^{d_l-1}\Phi\left(l^{d_l}/R^d\right)$$